J. Math. Anal. Appl. 383 (2011) 619–626

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On a class of singular p-Laplacian boundary value problems

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ARTICLE INFO

Article history:
Received 28 August 2010
Available online 30 May 2011
Submitted by P.J. McKenna

Keywords:
Positive solutions
Singular BVP
p-Laplacian

ABSTRACT

We prove the existence and nonexistence of positive solutions for the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $f : (0,\infty) \to \mathbb{R}$ is $(p-1)$-subhomogeneous at ∞ and is allowed to change sign, and λ is a large parameter. Our approach is based on the method of sub- and supersolutions.

1. Introduction

Consider the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{I}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, $f : (0,\infty) \to \mathbb{R}$, and λ is a positive parameter.

Singular boundary value problems of the type (I) have been studied extensively in recent years (see [2,3,5–11,15,17,18, 20] and the references therein). Problem (I) was considered in [6,8,9,15,20] in the case when f is $(p-1)$-subhomogeneous at ∞ and becomes $-\infty$ at 0. In [15], Ramaswamy et al. established positive solutions to the problem

$$\begin{cases} -\Delta_p u = \lambda\left(-\dfrac{1}{u^\beta} + f(u)\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for λ large when $p = 2$, $0 < \beta < 1$, $f : [0,\infty) \to \mathbb{R}$ is continuous, nondecreasing, $f(0) \geqslant 0$, and $\lim_{u\to\infty} f(u) = \infty$. In [20], Zhang proved the existence of $\bar\lambda > 0$ such that the problem

$$\begin{cases} -\Delta_p u = -\dfrac{1}{u^\beta} + \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

with $p = 2$, $f(u) = u^q$, $0 < \beta, q < 1$ has a positive solution for $\lambda > \bar\lambda$ and no positive solutions for $\lambda < \bar\lambda$. Related results can be found in [6,8,9]. In this paper, we shall obtain positive solutions to (I) when f is $(p-1)$-subhomogeneous at ∞ without requiring monotonicity or positivity conditions on f. In particular, our results when applied to (1.1) and (1.2) with

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.05.060

f continuous at 0 give the conclusion in [15,20] when f merely satisfies $f(u) \geqslant \frac{a}{u^\beta}$, where $a > 1$ in (1.1) and $a > 0$ in (1.2). It should be noted that, when applied to (1.1), we do not need $\beta < 1/n$ or $f(u)$ be bounded away from 0 for u large as in [8,9]. Our results extend and complement corresponding results in [8,9,15,20] in several ways. Our approach is based on the method of sub- and supersolutions.

2. Main results

We make the following assumptions:

(A.1) $f : (0, \infty) \to \mathbb{R}$ is continuous and

$$\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = 0.$$

(A.2) There exist positive numbers a, β, A with $\beta < 1$ such that

$$f(u) \geqslant \frac{a}{u^\beta} \quad \forall u > A,$$

and

$$\limsup_{u\to 0^+} u^\beta \left|f(u)\right| < \infty.$$

By a solution of (I), we mean a function $u \in C^1(\Omega) \cap C(\bar{\Omega})$ which satisfies (I) in the weak sense i.e.,

$$\int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \xi \, dx = \int_\Omega f(u) \xi \, dx$$

for all $\xi \in C_c^\infty(\Omega)$. Under appropriate conditions, it can be shown that such a solution belongs to $C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ (see Appendix A).

Theorem 2.1. *Let* (A.1)–(A.2) *hold. Then problem* (I) *has a positive solution u_λ for λ large. Furthermore $u_\lambda \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ and $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$. If, in addition, f is negative near 0 and nondecreasing then there exists a positive number λ_0 such that* (I) *has a positive solution for $\lambda > \lambda_0$ and no positive solutions for $\lambda < \lambda_0$.*

Corollary 2.1. *Let f satisfy* (A.1)–(A.2)*. Then problem* (1.2) *has a positive solution $u_\lambda \in C^{1,\alpha}(\bar{\Omega})$ for λ large and no positive solution for λ small. In addition, if f is nondecreasing then there exists a positive number λ_0 such that* (1.2) *has a positive solution for $\lambda > \lambda_0$ and no positive solutions for $\lambda < \lambda_0$.*

Example 2.1. Let $\gamma, \beta \in (0, 1)$ with $\gamma < \beta$ and $f(u) = -\frac{1}{u^\beta} + \frac{1}{u^\gamma}$. Then f satisfies (A.1), (A.2) and therefore (I) has a $C^{1,\alpha}(\bar{\Omega})$ positive solution for λ large.

3. Preliminary results

We shall denote the norms in $L^p(\Omega)$, $C^1(\bar{\Omega})$, and $C^{1,\alpha}(\bar{\Omega})$ by $\|\cdot\|_p$, $|\,.\,|_1$, and $|\,.\,|_{1,\alpha}$ respectively.

We first establish a regularity result, which plays a crucial role in the proof of the main results. Let $d(x)$ denote the distance from x to the boundary of Ω.

Lemma 3.1. *Let $h \in L^\infty_{loc}(\Omega)$ and suppose there exist numbers $\gamma \in (0, 1)$ and $C > 0$ such that*

$$\left|h(x)\right| \leqslant \frac{C}{d^\gamma(x)} \tag{3.1}$$

for a.e. $x \in \Omega$. Let $u \in W_0^{1,p}(\Omega)$ be the solution of

$$\begin{cases} -\Delta_p u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Then there exist constants $\alpha \in (0, 1)$ and $M > 0$ depending only on C, γ, Ω such that $u \in C^{1,\alpha}(\bar{\Omega})$ and $|u|_{1,\alpha} < M$.

Proof. Note that Lemma 3.1 was proved in [7] under the additional assumptions that $h \geqslant 0$ and $u \leqslant \tilde{C} d$ in Ω for some $\tilde{C} > 0$. When $p = 2$, it was established in [10]. For convenience, we give a proof.

Suppose $p = 2$. It follows from [3] that the problem

$$-\Delta v = \frac{1}{v^\gamma} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega$$

has a positive solution v which is Lipschitz continuous in $\bar{\Omega}$. Let $C_1 > 0$ be such that $v(x) \leqslant C_1 d(x)$ in Ω. Then

$$-\Delta\big(CC_1^\gamma v\big) \geqslant \frac{C}{d^\gamma} \quad \text{in } \Omega.$$

Let $\tilde{u}$ be the solution of

$$-\Delta\tilde{u} = |h| \quad \text{in } \Omega, \qquad \tilde{u} = 0 \quad \text{on } \partial\Omega,$$

and $\bar{u} = u + \tilde{u}$. Then

$$-\Delta\bar{u} = h + |h| \geqslant 0 \quad \text{in } \Omega.$$

By the maximum principle, $\tilde{u}(x) \leqslant CC_1^\gamma v(x) \leqslant C_2 d(x)$ and $\bar{u}(x) \leqslant 2C_2 d(x)$ for $x \in \Omega$. Using the regularity result in [7, Theorem B.1], we conclude that there exist $\alpha \in (0,1)$ and $M_0 > 0$ such that $\tilde{u}, \bar{u} \in C^{1,\alpha}(\bar{\Omega})$ and $|\tilde{u}|_{1,\alpha}, |\bar{u}|_{1,\alpha} < M_0$. Since $u = \bar{u} - \tilde{u}$, Lemma 3.1 with $p = 2$ follows.

Now let u be the solution of (3.2) with $p > 1$. From Lemma 3.1, Theorem B.1, and the proof of Lemma A.7 in [7], it follows that the problem

$$\begin{cases} -\Delta_p v = \dfrac{C}{v^\gamma} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique positive solution $v \in W_0^{1,p}(\Omega)$ with $v \leqslant c_0 d$ in Ω. This implies

$$-\Delta_p\big(c_0^{\frac{\gamma}{p-1}} v\big) \geqslant \frac{C}{d^\gamma} \quad \text{in } \Omega.$$

Since

$$-\Delta_p u \leqslant \frac{C}{d^\gamma} \quad \text{and} \quad -\Delta_p(-u) \leqslant \frac{C}{d^\gamma}$$

in Ω, the weak comparison principle (see e.g. [16, Lemma A.2]) implies

$$|u| \leqslant c_0^{\frac{\gamma}{p-1}} v \leqslant c_0^{\frac{\gamma}{p-1}+1} d \quad \text{in } \Omega.$$

Next, let $w \in C^{1,\alpha}(\bar{\Omega})$ be the solution of

$$-\Delta w = h \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

Then

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u - \nabla w\big) = 0 \quad \text{in } \Omega,$$

and Lemma 3.1 now follows from Lieberman's result [13, Theorem 1]. □

Corollary 3.1. *Let $\varepsilon > 0$ and $h, \tilde{h} \in L^\infty_{loc}(\Omega)$ satisfy* (3.1) *with $h \geqslant 0$, $h \not\equiv 0$. Let $u, u_\varepsilon \in W_0^{1,p}(\Omega)$ satisfy*

$$\begin{cases} -\Delta_p u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$-\Delta_p u_\varepsilon = \begin{cases} h & \text{if } d(x) > \varepsilon, \\ \tilde{h} & \text{if } d(x) < \varepsilon. \end{cases}$$

Then for ε small enough,

$$u_\varepsilon \geqslant u/2 \quad \text{in } \Omega.$$

Proof. By Lemma 3.1, there exist $M > 0$ and $\alpha \in (0,1)$ such that $|u|_{1,\alpha}, |u_\varepsilon|_{1,\alpha} < M$. By the strong maximum principle [19], there exists $\kappa > 0$ such that $u \geqslant \kappa d$ in Ω. Multiplying the equation

$$-\Delta_p u - (-\Delta_p u_\varepsilon) = \begin{cases} 0 & \text{if } d(x) > \varepsilon, \\ h - \tilde{h} & \text{if } d(x) < \varepsilon \end{cases}$$

by $u - u_\varepsilon$ and integrating gives

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right).\nabla(u - u_\varepsilon)\,dx \leqslant 2M \int_{d<\varepsilon} |h - \tilde{h}|\,dx.$$

Note that for $x, y \in \mathbb{R}^n$,

$$\left(|x| + |y|\right)^{2-\min(p,2)}\left(|x|^{p-2}x - |y|^{p-2}y\right).(x-y) \geqslant C_0|x-y|^{\max(p,2)},$$

where $C_0 = (1/2)^{p-1}$ if $p \geqslant 2$, $C_0 = p-1$ if $p < 2$ (see e.g. [14, Lemma 30.1]).

Hence $\|\nabla(u - u_\varepsilon)\|_2 \to 0$ as $\varepsilon \to 0$, and since $C^{1,\alpha}(\bar{\Omega})$ is compactly imbedded in $C^1(\bar{\Omega})$, we obtain $|u - u_\varepsilon|_1 \to 0$ as $\varepsilon \to 0$. Consequently, if ε is sufficiently small,

$$|u_\varepsilon - u|_1 \leqslant \kappa/2,$$

which implies

$$u_\varepsilon \geqslant u - (\kappa/2)d \geqslant u/2 \quad \text{in } \Omega,$$

which completes the proof. □

4. Proofs of main results

We are now ready to give the proofs of the main results. We shall denote by ψ the solution of

$$\begin{cases} -\Delta_p \psi = 1 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof of Theorem 2.1. By (A.1), (A.2), there exists $b > 0$ such that

$$f(u) \geqslant -\frac{b}{u^\beta}$$

for $u > 0$. Let $\varepsilon > 0$ and ϕ be the positive solution of

$$\begin{cases} -\Delta_p \phi = \dfrac{1}{\phi^\beta} & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

with $\phi \geqslant k_1 d$ in Ω for some $k_1 > 0$ (see Lemma 3.1 in [7]). Let ϕ_0 satisfy

$$-\Delta_p \phi_0 = \begin{cases} \frac{\delta}{\phi^\beta} & \text{if } d(x) > \varepsilon, \\ -\frac{k}{\phi^\beta} & \text{if } d(x) < \varepsilon, \end{cases} \qquad \phi_0 = 0 \quad \text{on } \partial\Omega,$$

where $k = 2^\beta b \delta^{-\frac{\beta}{p-1}}$ and $\delta > 0$ is such that $\delta^{1+\beta/(p-1)} = a$. By the weak comparison principle, $\phi_0 \leqslant \delta^{\frac{1}{p-1}}\phi$ in Ω. Using Corollary 3.1, we obtain $\phi_0 \geqslant (\delta^{\frac{1}{p-1}}/2)\phi$ in Ω if ε is sufficiently small, which we shall assume.

We shall verify that $\Phi = \lambda^r \phi_0$, where $r = \frac{1}{\beta+p-1}$, is a subsolution of (I). Let $\xi \in W_0^{1,p}(\Omega)$ with $\xi \geqslant 0$. Then

$$\int_\Omega |\nabla\Phi|^{p-2}\nabla\Phi.\nabla\xi\,dx = \lambda^{r(p-1)}\delta \int_{d>\varepsilon} \frac{\xi}{\phi^\beta}\,dx - \lambda^{r(p-1)}k \int_{d<\varepsilon} \frac{\xi}{\phi^\beta}\,dx. \tag{4.1}$$

For $d(x) > \varepsilon$, we have

$$\Phi(x) \geqslant \lambda^r\left(\delta^{\frac{1}{p-1}}/2\right)\phi(x) \geqslant \lambda^r\left(\delta^{\frac{1}{p-1}}/2\right)k_1 d(x) > \lambda^r\left(\delta^{\frac{1}{p-1}}/2\right)k_1\varepsilon > A$$

if λ is large enough. Hence

$$\lambda \int_{d>\varepsilon} f(\Phi)\xi\,dx \geqslant \lambda a \int_{d>\varepsilon} \frac{\xi}{\Phi^\beta}\,dx \geqslant \lambda^{1-r\beta} a \delta^{-\frac{\beta}{p-1}} \int_{d>\varepsilon} \frac{\xi}{\phi^\beta}\,dx = \lambda^{r(p-1)}\delta \int_{d>\varepsilon} \frac{\xi}{\phi^\beta}\,dx. \tag{4.2}$$

On the other hand,

$$\lambda \int\limits_{d<\varepsilon} f(\Phi)\xi\,dx \geqslant -\lambda b \int\limits_{d<\varepsilon} \frac{\xi}{\Phi^\beta}\,dx \geqslant -\lambda^{r(p-1)}k \int\limits_{d<\varepsilon} \frac{\xi}{\phi^\beta}\,dx. \tag{4.3}$$

Combining (4.1)–(4.3), we see that

$$\int\limits_\Omega |\nabla\Phi|^{p-2}\nabla\Phi.\nabla\xi\,dx \leqslant \lambda \int\limits_\Omega f(\Phi)\xi\,dx,$$

i.e., Φ is a subsolution of (I). Next, we shall create a supersolution Ψ of (I) with $\Psi \geqslant \Phi$ in Ω. Let λ be as in the above and choose $\varepsilon_0 > 0$ so that $\lambda\varepsilon_0\|\phi\|_\infty^{p-1+\beta} < 1$. By (A.1), there exists $A_1 > 0$ such that

$$\left|f(u)\right| \leqslant \varepsilon_0 u^{p-1}$$

for $u > A_1$. Since $\limsup_{u\to 0^+} u^\beta|f(u)| < \infty$, there exists $C > 0$ such that

$$\left|f(u)\right| \leqslant \frac{C}{u^\beta}$$

for $u \leqslant A_1$. Choose $M > \lambda^r\delta^{\frac{1}{p-1}}$ so that

$$\lambda\varepsilon_0\big(M\|\phi\|_\infty\big)^{p-1+\beta} + \lambda C \leqslant M^{p-1+\beta}$$

and let $\Psi = M\phi$. Then $\Psi \geqslant \Phi$ in Ω and we have

$$\lambda f(M\phi) \leqslant \lambda\left[\varepsilon_0(M\phi)^{p-1} + \frac{C}{(M\phi)^\beta}\right] \leqslant \frac{\lambda\varepsilon_0(M\|\phi\|_\infty)^{p-1+\beta} + \lambda C}{M^\beta\phi^\beta} \leqslant \frac{M^{p-1}}{\phi^\beta},$$

which implies

$$\lambda \int\limits_\Omega f(\Psi)\xi\,dx \leqslant M^{p-1} \int\limits_\Omega \frac{\xi}{\phi^\beta}\,dx = \int\limits_\Omega |\nabla\Psi|^{p-2}\nabla\Psi.\nabla\xi\,dx,$$

i.e., Ψ is a supersolution of (I). By Lemma A in Appendix A, (I) has a solution $u_\lambda \in C^{1,\alpha}(\bar\Omega)$ with $\Phi \leqslant u_\lambda \leqslant \Psi$ in Ω. In particular, $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$. Suppose next that f is nondecreasing and negative near 0. Then there exists $M_0 > 0$ such that

$$f(u) \leqslant M_0 u^{p-1}$$

for $u > 0$. Let u be a positive solution of (I). Then

$$-\Delta_p u \leqslant \lambda M_0 u^{p-1} \leqslant \lambda M_0\|u\|_\infty^{p-1},$$

which implies

$$u \leqslant (\lambda M_0)^{\frac{1}{p-1}}\|u\|_\infty\psi \quad \text{in } \Omega,$$

and so $u = 0$ if $(\lambda M_0)^{\frac{1}{p-1}}\|\psi\|_\infty < 1$, i.e., (I) has no positive solutions for λ small. Next, we note that u is a solution of (I) if and only if $v = \lambda^{-\frac{1}{p-1}}u$ is a solution of

$$\begin{cases} -\Delta_p v = f\big(\lambda^{\frac{1}{p-1}}v\big) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{I_λ}$$

Suppose $\lambda < \tilde\lambda$ and v_λ is a positive solution of (I_λ) in $C^1(\bar\Omega)$. Since f is nondecreasing, v_λ is a subsolution of $(\mathrm{I}_{\tilde\lambda})$. As in the above, there exists a supersolution Ψ of $(\mathrm{I}_{\tilde\lambda})$ such that $\Psi \geqslant v_\lambda$ in Ω. Hence $(\mathrm{I}_{\tilde\lambda})$ has a solution $v_{\tilde\lambda}$ with $\Psi \geqslant v_{\tilde\lambda} \geqslant v_\lambda$ in Ω (see [12, Lemma 1.8]). Let $\lambda_0 = \inf\{\lambda\colon\ (\mathrm{I}_\lambda) \text{ has a positive solution}\}$. Then (I_λ) has a positive solution for $\lambda > \lambda_0$ and no positive solutions for $\lambda < \lambda_0$. □

Proof of Corollary 2.1. By Theorem 2.1, the problem

$$\begin{cases} -\Delta_p u = \lambda\left(-\frac{1}{u^\beta} + f(u)\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a positive solution u for λ large. For $\lambda > 1$, we have

$$\lambda\left(-\frac{1}{u^\beta} + f(u)\right) \leqslant -\frac{1}{u^\beta} + \lambda f(u) \quad \text{in } \Omega,$$

and hence u is a subsolution of (1.2). As in the proof of Theorem 2.1, we obtain a supersolution Ψ of (1.2) with $\Psi \geqslant u$ in Ω. Hence (1.2) has a positive solution for λ large. By (A.1)–(A.2), there exist $M_1, M_2 > 0$ such that

$$f(u) \leqslant \frac{M_1}{u^\beta} + M_2 u^{p-1}$$

for $u > 0$. Hence if u is a positive solution of (1.2), we have

$$-\Delta_p u \leqslant -\frac{1}{u^\beta} + \frac{\lambda M_1}{u^\beta} + \lambda M_2 u^{p-1} \leqslant \lambda M_2 u^{p-1} \quad \text{in } \Omega,$$

if $\lambda M_1 < 1$. This implies $u = 0$ if λ is sufficiently small. Finally, note that u is a solution of (1.2) if and only if $v = \lambda^{-\frac{1}{p-1}} u$ is a solution of

$$\begin{cases} -\Delta_p v = -\dfrac{\lambda^{-\frac{p-1+\beta}{p-1}}}{v^\beta} + f\left(\lambda^{\frac{1}{p-1}} v\right) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence if f is nondecreasing then a solution of (1.2_λ) is a subsolution of $(1.2_{\tilde{\lambda}})$ for $\tilde{\lambda} > \lambda$, from which the existence of λ_0 follows. □

Acknowledgment

The author thanks Prof. Peter Takáč for stimulating discussions during his visit to Mississippi State University in spring 2009.

Appendix A

We shall present some results in sub- and supersolutions method for singular boundary value problems. Related results can be found in [4]. Consider the problem

$$\begin{cases} -\Delta_p u = h(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

where $h : \Omega \times (0, \infty) \to \mathbb{R}$ is continuous.

Let $\phi, \psi \in C^1(\bar{\Omega})$ satisfy $\phi, \psi \geqslant ld$ in Ω for some $l > 0$ and suppose there exist $\gamma \in (0, 1)$ and $C > 0$ such that

$$\left|h(x, w)\right| \leqslant \frac{C}{d^\gamma(x)}$$

for a.e. $x \in \Omega$ and all $w \in C(\bar{\Omega})$ with $\phi \leqslant w \leqslant \psi$ in Ω. Suppose ϕ, ψ are sub- and supersolutions of (A.1) respectively i.e., for all $\xi \in W_0^{1,p}(\Omega)$ with $\xi \geqslant 0$,

$$\int_\Omega |\nabla\phi|^{p-2}\nabla\phi.\nabla\xi \, dx \leqslant \int_\Omega h(x, \phi)\xi \, dx,$$

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi.\nabla\xi \, dx \geqslant \int_\Omega h(x, \psi)\xi \, dx,$$

and $\phi \leqslant 0 \leqslant \psi$ on $\partial\Omega$. Note that the integrals on the right-hand side are defined by virtue of Hardy's inequality (see e.g. [1]).

Lemma A. *Under the above assumptions, there exists $\alpha \in (0, 1)$ such that* (A.1) *has a solution $u \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$.*

Proof. For each $v \in C(\bar{\Omega})$, let $u = Tv$ be the unique solution of

$$\begin{cases} -\Delta_p u = \tilde{h}(x, v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{h}(x, v) = h(x, \tilde{v})$, $\tilde{v} = \min(\max(v, \phi), \psi)$. Note that $\phi \leqslant \tilde{v} \leqslant \psi$ in Ω. Since $|\tilde{h}(x, v)| \leqslant \frac{C}{d^\gamma(x)}$ for a.e. $x \in \Omega$ and all $v \in C(\bar{\Omega})$, Lemma 3.1 implies the existence of $\alpha \in (0, 1)$ such that $u \in C^{1,\alpha}(\bar{\Omega})$ and $|u|_{1,\alpha} < \tilde{C}$, where $\tilde{C}$ is independent of v.

Since $C^{1,\alpha}(\bar{\Omega})$ is compactly imbedded in $C(\bar{\Omega})$, it follows that $T(C(\bar{\Omega}))$ is relatively compact in $C(\bar{\Omega})$. We shall verify that T is continuous. Let $v \in C(\bar{\Omega})$ and (v_n) be a sequence in $C(\bar{\Omega})$ which converges to v in $C(\bar{\Omega})$. Let $u_n = Tv_n$ and $u = Tv$. Then we have

$$-\Delta_p u_n - (-\Delta_p u) = \tilde{h}(x, v_n) - \tilde{h}(x, v) \quad \text{in } \Omega,$$

which implies, after multiplying by $u_n - u$ and integrating,

$$\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot \nabla(u_n - u)\,dx$$
$$= \int_\Omega \left(\tilde{h}(x, v_n) - \tilde{h}(x, v)\right)(u_n - u)\,dx \leqslant 2\tilde{C}\int_\Omega \left|\tilde{h}(x, v_n) - \tilde{h}(x, v)\right| dx.$$

By Lemma 30.1 in [14], there exists a positive number C_0 such that for $x, y \in \mathbb{R}^n$,

$$\left(|x| + |y|\right)^{2-\min(p,2)}\left(|x|^{p-2}x - |y|^{p-2}y\right)\cdot(x - y) \geqslant C_0|x - y|^{\max(p,2)}.$$

Hence there exists $C_1 > 0$ such that

$$\int_\Omega \left|\nabla(u_n - u)\right|^{\max(p,2)} dx \leqslant C_1\int_\Omega \left|\tilde{h}(x, v_n) - \tilde{h}(x, v)\right| dx.$$

Since $\tilde{h}(x, v_n(x)) \to \tilde{h}(x, v(x))$ for all $x \in \Omega$, and

$$\left|\tilde{h}(x, v_n)\right| \leqslant \frac{C}{d^\gamma(x)}$$

for a.e. $x \in \Omega$ and all n, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\int_\Omega \left|\tilde{h}(x, v_n) - \tilde{h}(x, v)\right| dx = 0,$$

and so $u_n \to u$ in $H_0^1(\Omega)$. This, together with the fact that (u_n) is bounded in $C^{1,\alpha}(\bar{\Omega})$, implies $u_n \to u$ in $C^1(\bar{\Omega})$. Hence $T : C(\bar{\Omega}) \to C(\bar{\Omega})$ is continuous. Hence, by the Schauder Fixed Point Theorem, T has a fixed point u. We shall verify that $\phi \leqslant u \leqslant \psi$ in Ω. Let $\xi = (\phi - u)^+$ and suppose $\xi \not\equiv 0$. Then

$$\int_{u<\phi} |\nabla u|^{p-2}\nabla u\cdot\nabla\xi\,dx = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\xi\,dx = \int_{u<\phi}\tilde{h}(x, u)\xi\,dx$$
$$= \int_{u<\phi} h(x, \phi)\xi\,dx \geqslant \int_{u<\phi} |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla\xi\,dx,$$

which implies

$$\int_{u<\phi}\left(|\nabla u|^{p-2}\nabla u - |\nabla\phi|^{p-2}\nabla\phi\right)\cdot\nabla(u - \phi)\,dx \leqslant 0,$$

a contradiction. Hence $u \geqslant \phi$ in Ω, and similarly we get $u \leqslant \psi$ in Ω. By Lemma 3.1, $u \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$. □

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